JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **156**, 293–304 (1991)

Boundedness of Solutions of a Nonlinear Nonautonomous Neutral Delay Equation

YANG KUANG* AND ALAN FELDSTEIN

Department of Mathematics, Arizona State University, Tempe, Arizona 85287-1804

Submitted by V. Lakshmikantham

Received June 8, 1989

Sufficient conditions are obtained for the boundedness of solutions of the nonlinear nonautonomous neutral equation

$$\dot{x}(t)=r(t)\,x(t)(a(t)-x(t-1)-c(t)\,\dot{x}(t-1)),$$

which arise in a "food-limited" population model. This partially answers a recent open question proposed by K. Gopalsamy and B. G. Zhang. © 1991 Academic Press, Inc.

1. INTRODUCTION

The nonlinear neutral delay logistic equation was first introduced and extensively discussed in [6],

$$\frac{dx(t)}{dt}=\dot{x}(t)=rx(t)(1-(x(t-\tau)+c\dot{x}(t-\tau))/K), \tag{1.1}$$

where r, τ, c, K are assumed to be positive constants. This equation is a natural generalization of the well-known elementary logistic equation which has been used as a model for single species population growth,

$$\dot{x}(t)=rx(t)(1-x(t)/K), \tag{1.2}$$

where r is the intrinsic growth rate of species x and K is interpreted as the environment capacity for x. In [11, pp. 38–40], F. E. Smith argued that per capita growth rate $r(1-x(t)/K)$ in (1.2) should be replaced by $r(1-(x(t)+c\dot{x}(t))/K)$, based on his investigation on laboratory populations of Daphnia magna. Obviously, it will be even more realistic to incor-

* The research of this author was partially supported by a Faculty Grant-in-Aid Award and a CLAS Summer Research Award at Arizona State University.

0022-247X/91 $3.00

409/156/1-20

porate a single discrete delay τ in the per capita growth rate, which results in (1.1). The detailed ecological justification of Eq. (1.1) can be found in [6] and [11]. In [6], sufficient conditions are obtained for the local asymptotic stability of the positive steady state of (1.1). The oscillatory and nonoscillatory behavior of the positive solutions of (1.1) are also studied in [6]. In [3], necessary and sufficient conditions are found for the local stability and stability switches of the positive steady state of (1.1), which improves one of the results in [6].

One of the most important questions left unresolved in paper [6] is whether or not all solutions of (1.1) corresponding to a suitable class of positive initial values remain bounded as $t \to \infty$. It is well known that if $c = 0$ in (1.1), all positive solutions remain bounded as $t \to \infty$. The main purpose of this paper is to show that indeed there are reasonable conditions under which solutions of (1.1) will remain bounded as $t \to \infty$. In fact, the results are valid for the more general nonautonomous version of (1.1),

$$\dot{x}(t) = r(t)\, x(t)(a(t) - b(t)\, x(t-\tau) - c(t)\, \dot{x}(t-\tau)), \tag{1.3}$$

where $r(t) > 0$ and $b(t) > 0$. Obviously, it is easier to work with equations that have less variables and parameters. Thus, take

$$t = \tau s, \qquad \bar{x}(s) = x(\tau s),$$

$$\bar{r}(s) = \tau r(\tau s)\, b(\tau(s)), \qquad \bar{a}(s) = \frac{a(\tau s)}{b(\tau s)}, \qquad \bar{c}(s) = \frac{1}{\tau}\frac{c(\tau s)}{b(\tau s)}.$$

With these scalings, (1.3) becomes

$$\frac{d\bar{x}(s)}{ds} = \bar{r}(s)\, \bar{x}(s) \left(\bar{a}(s) - \bar{x}(s-1) - \bar{c}(s) \frac{d\bar{x}(s-1)}{ds} \right). \tag{1.4}$$

Therefore, the rest of this paper discusses the equation

$$\dot{x}(t) = r(t)\, x(t)(a(t) - x(t-1) - c(t)\, \dot{x}(t-1)), \tag{1.5}$$

rather than Eq. (1.3). Equation (1.5) *always assumes that* $r(t)$ *is positive, that* $a(t) \cdot c(t) \neq 0$, *and that all of these three functions are continuous and bounded both from below and above.* The approach throughout this paper is mainly motivated by the intrinsic interest of the equation and the ecological interpretation of the results.

The next section contains nonboundedness results. Section 3, the main part of this paper, contains the boundedness results. It is followed by a section devoted to the discussion of the constant coefficients case; results there are compared with an asymptotic result obtained in [3]. The final section consists of a brief list of questions remaining to be investigated.

2. Nonboundedness Results

This section presents conditions under which the positive solutions of (1.5) indeed tend to infinity as $t \to \infty$. Before stating the related theorems, the following result is needed.

LEMMA 2.1. *Consider Eq.* (1.5). *If* $x(0) > 0$, *then* $x(t) > 0$ *for all* $t \geqslant 0$. *If* $x(0) = 0$, *then* $x(t) \equiv 0$ *for all* $t \geqslant 0$.

Proof. Since $x(0) > 0$, for t close to 0, $x(t) > 0$; therefore, Eq. (1.5) can be written as

$$d(\ln x(t))/dt = r(t)(a(t) - x(t-1) - c(t)\,\dot{x}(t-1)).$$

This implies

$$x(t) = x(0) \exp\left(\int_0^t r(s)(a(s) - x(s-1) - c(s)\,\dot{x}(s-1)\,ds\right). \tag{2.1}$$

Obviously, $x(t) > 0$ for $t \in [0, 1]$, in particular, $x(1) > 0$. By induction, it follows that $x(t) > 0$ for all $t \geqslant 0$. If $x(0) = 0$, (2.1) indicates that $x(t) \equiv 0$, for $t \in [0, 1]$. Again, by induction, $x(t) \equiv 0$, for all $t \geqslant 0$. ∎

Note that Eq. (1.5) can be written as

$$\dot{x}(t) - r(t)\,a(t)\,x(t) = -r(t)\,x(t)(c(t)\,\dot{x}(t-1) + x(t-1)). \tag{2.2}$$

Since $c(t) \neq 0$ for all $t \geqslant 0$, then Eq. (2.2) is equivalent to

$$\dot{x}(t) - r(t)\,a(t)\,x(t) = -c(t)\,r(t)\,x(t)(\dot{x}(t-1) + c^{-1}(t)\,x(t-1)). \tag{2.3}$$

This expression leads to the following theorem.

THEOREM 2.1. *In Eq.* (1.5), *assume* $r(t) > 0$, $a(t) > 0$, $-c(t) > 0$, $c^{-1}(t) = -r(t-1)a(t-1)$, *for* $t \geqslant 1$ *and* $x(0) > 0$.

(1) *If* $\dot{x}(t) + c^{-1}(t)\,x(t) \geqslant 0$ *for* $-1 \leqslant t \leqslant 0$, *then* $x(t)$ *is unbounded. In fact,* $x(t) \geqslant x(0) \exp(\int_0^t r(s)\,a(s)\,ds)$.

(2) *If* $\dot{x}(t) + c^{-1}(t)\,x(t) \leqslant 0$ *for* $-1 \leqslant t \leqslant 0$, *then* $x(t) \leqslant x(0) \exp(\int_0^t r(s)\,a(s)\,ds)$ *for* $t \geqslant 0$.

Proof. By Lemma 2.1, $x(t) > 0$; hence, $-c(t)\,r(t)\,x(t) > 0$, for $t \geqslant 0$. Denote

$$F(t) = \dot{x}(t) - r(t)\,a(t)\,x(t) \qquad \text{for} \quad t \geqslant 0. \tag{2.4}$$

Equation (2.3) indicates that $F(t)$ will not change sign for $t \geqslant 0$ if $\dot{x}(t) + c^{-1}(t)\,x(t)$ does not change sign in the interval $[-1, 0]$. In other words, if $\dot{x}(t)+c^{-1}(t)\,x(t)\geqslant 0$, for $-1\leqslant t\leqslant 0$, then $F(t)\geqslant 0$, for all $t\geqslant 0$; therefore,

$$\dot{x}(t)-r(t)\,a(t)\,x(t)\geqslant 0 \qquad \text{for} \quad t\geqslant 0.$$

This implies that

$$x(t)\geqslant x(0)\exp\left(\int_0^t r(s)\,a(s)\,ds\right).$$

This proves case (1). On the other hand, if

$$\dot{x}(t)-r(t)\,a(t)\,x(t)\leqslant 0 \qquad \text{for} \quad t\geqslant 0,$$

then

$$0<x(t)\leqslant x(0)\exp\left(\int_0^t r(s)\,a(s)\,ds\right),$$

which implies the conclusion in (2). ∎

Motivated by Theorem 2.1, the following more general result can be proved.

THEOREM 2.2. *In Eq.* (1.5), *assume* $r(t)>0$, $a(t)>0$, $c(t)<0$, *and* $c^{-1}(t)=-r(t-1)a(t-1)+\varepsilon(t)$, $x(t)>0$, *for* $t\in[0,1]$.

(1) *If* $\varepsilon(t)\geqslant 0$ *and* $\dot{x}(t)-r(t)\,a(t)\,x(t)\geqslant 0$ *for* $-1\leqslant t\leqslant 0$, *then* $x(t)\geqslant x(0)\exp(\int_0^t r(s)\,a(s)\,ds)$.

(2) *If* $\varepsilon(t)\leqslant 0$ *and* $\dot{x}(t)-r(t)\,a(t)\,x(t)\leqslant 0$ *for* $-1\leqslant t\leqslant 0$, *then* $x(t)\leqslant x(0)\exp(\int_0^t r(s)\,a(s)\,ds)$, *for* $t\geqslant 0$.

Proof. Equation (2.3) yields

$$\begin{aligned}\dot{x}(t)-r(t)\,a(t)\,x(t)=(\varepsilon(t)-r(t-1)\,a(t-1))^{-1}\,r(t)\,x(t)(\dot{x}(t-1)\\ -r(t-1)\,a(t-1)\,x(t-1)+\varepsilon(t)\,x(t-1)).\end{aligned} \tag{2.5}$$

There are two cases to consider. If $\varepsilon(t)\geqslant 0$, then

$$\begin{aligned}\dot{x}(t)-r(t)\,a(t)\,x(t)\geqslant(\varepsilon(t)-r(t-1)\,a(t-1))^{-1}\,r(t)\,x(t)\\ \times(\dot{x}(t-1)-r(t-1)\,a(t-1)\,x(t-1)),\end{aligned}$$

because $x(t)>0$ by Lemma 2.1 and because it was assumed that $c(t)<0$, and that $r(t)>0$. Obviously, $\dot{x}(t)-r(t)\,a(t)\,x(t)\geqslant 0$, for $-1\leqslant t\leqslant 0$ ensures that $\dot{x}(t)-r(t)\,a(t)\,x(t)\geqslant 0$ for $0\leqslant t\leqslant 1$. Induction immediately yields the conclusion that $x(t)\geqslant x(0)\exp(\int_0^t r(s)\,a(s)\,ds)$, for $t\geqslant 0$.

If $\varepsilon(t) \leqslant 0$, Eq. (2.5) implies that

$$\dot{x}(t) - r(t)\,a(t)\,x(t) \leqslant (\varepsilon(t) - r(t-1)\,a(t-1))^{-1}\,r(t)\,x(t) \\ \times (\dot{x}(t-1) - r(t-1)\,a(t-1)\,x(t-1)).$$

A similar argument as in case of (1), yields

$$x(t) \leqslant x(0) \exp\left(\int_0^t r(s)\,a(s)\,ds\right) \qquad \text{for} \quad t \geqslant 0. \quad \blacksquare$$

The assumption in this section that $c(t) < 0$ is contrary to the assumption made in [6] and [11]. However, this may well be a reasonable hypothesis for some species. For example, if an immature organism consumes less food than a mature one, then $c(t)$ should be negative rather than positive. If $c(t) = 0$, then all solutions of (1.5) are eventually bounded as long as both $r(t)$ and $a(t)$ are positive and bounded from below and above. Thus, the results in this section indicate that the neutral term in Eq. (1.5) is destabilizing.

3. Boundedness Results

This section obtains conditions under which solutions of Eq. (1.5) will be bounded. Throughout this section, it is assumed that there are finite constants r_1, r_2, c_1, and c_2, such that $0 < r_1 \leqslant r(t) \leqslant r_2$, and $0 < c_1 \leqslant c(t) \leqslant c_2$.

Equation (1.5) may be rewritten as

$$\dot{x}(t) + c^{-1}(t+1)\,x(t) = r(t)\,x(t)\{a(t) + r^{-1}(t)\,c^{-1}(t+1) \\ - c(t)[\dot{x}(t-1) + c^{-1}(t)\,x(t-1)]\}. \qquad (3.1)$$

Denote

$$y(t) = \dot{x}(t) + c^{-1}(t+1)\,x(t). \qquad (3.2)$$

Then (3.1) becomes

$$y(t) = r(t)\,c(t)\,x(t)(c^{-1}(t)(a(t) + r^{-1}(t)\,c^{-1}(t+1)) - y(t-1)). \quad (3.3)$$

Denote

$$P(t) = r^{-1}(t)\,c^{-1}(t), \qquad Q(t) = c^{-1}(t)(a(t) + r^{-1}(t)\,c^{-1}(t+1)). \qquad (3.4)$$

Assume there are positive constants P, Q_1, and Q_2, such that $P(t) \geqslant P$, $Q_1 \leqslant Q(t) \leqslant Q_2$.

LEMMA 3.1. *Let $T \geqslant 1$. For $t \leqslant T$, assume that the solution $x(t)$ of Eq. (1.5) satisfies $0 \leqslant x(t) \leqslant P(t)\, Q_1/Q_2$, and $0 \leqslant y(t) \leqslant Q_1$, for $-1 \leqslant t \leqslant 0$. Then,*

(1) $0 \leqslant y(t) \leqslant Q_1$, *for all* $0 \leqslant t \leqslant T$.

(2) $\int_{\min\{0, t-2\}}^{t} y(s)\, ds \leqslant Q_2$ *for* $0 \leqslant t \leqslant T$.

Proof. First assume $0 \leqslant t \leqslant 1$. Since $0 \leqslant y(t) \leqslant Q_1$ for $-1 \leqslant t \leqslant 0$ and $Q_1 \leqslant Q(t)$ for $-1 \leqslant t$, then Eq. (3.3) implies that $0 \leqslant y(t)$ for $0 \leqslant t \leqslant 1$. Hence, for $0 \leqslant t \leqslant 1$, Eqs. (3.3) and (3.4) plus $0 \leqslant x(t) \leqslant P(t)\, Q_1/Q_2$ and $Q(t) \leqslant Q_2$ together yield

$$\begin{aligned} y(t) &\leqslant x(t)\, P^{-1}(t)\, Q(t) \\ &\leqslant Q_1 Q(t)/Q_2 \leqslant Q_1. \end{aligned}$$

By induction, $0 \leqslant y(t) \leqslant Q_1$ for all $0 \leqslant t \leqslant T$. This establishes conclusion (1). Therefore,

$$\begin{aligned} \int_{t-1}^{t} y(s)\, ds &= \int_{t-1}^{t} x(s)\, P^{-1}(s)(Q(s) - y(s-1))\, ds \\ &\leqslant \int_{t-1}^{t} \frac{Q_1}{Q_2} (Q(s) - y(s-1))\, ds \\ &\leqslant \int_{t-1}^{t} (Q(s) - y(s-1))\, ds \\ &= \int_{t-1}^{t} Q(s)\, ds - \int_{t-1}^{t} y(s-1)\, ds \\ &\leqslant \int_{t-1}^{t} Q_2\, ds - \int_{t-2}^{t-1} y(s)\, ds. \end{aligned}$$

Hence,

$$\int_{t-2}^{t} y(s)\, ds \leqslant Q_2 \qquad \text{for} \quad 1 \leqslant t \leqslant T. \quad \blacksquare$$

Equation (3.2) yields

$$x(t) = x(0)\, e^{-\int_0^t c^{-1}(s+1)\, ds} + \int_0^t y(\sigma)\, e^{-\int_\sigma^t c^{-1}(s+1)\, ds}\, d\sigma. \tag{3.5}$$

LEMMA 3.2. *For $-1 \leqslant t \leqslant T$, assume that $y(t) \geqslant 0$ and that $\int_{\min\{0, t-2\}}^{t} y(s)\, ds \leqslant Q_2$. Then, $x(t) < x(0)\, e^{-\int_0^t c^{-1}(s+1)\, ds} + Q_2 e^{2c_2^{-1}}/(e^{2c_2^{-1}} - 1)$, for $t \in [0, T]$.*

Proof. Consider an estimate for

$$I=\int_0^t y(\sigma)\, e^{-\int_\sigma^t c^{-1}(s+1)\,ds}\, d\sigma,$$

since

$$I=\left(\int_{t-2}^t+\int_{t-4}^{t-2}+\cdots+\int_0^{t-2[t/2]}\right) y(\sigma)\, e^{-\int_\sigma^t c^{-1}(s+1)\,ds}\, d\sigma.$$

It is necessary to concentrate temporarily on the estimation of the integral on the interval $[t-2(n+1), t-2n]$. From the assumption $c^{-1}(t)\geqslant c_2^{-1}>0$, it follows that

$$-\int_\sigma^t c^{-1}(s+1)\,ds\leqslant -c_2^{-1}(t-\sigma).$$

Hence

$$\begin{aligned}\int_{t-2(n+1)}^{t-2n} & y(\sigma)\, e^{-\int_\sigma^t c^{-1}(s+1)\,ds}\, d\sigma \\ &\leqslant \int_{t-2(n+1)}^{t-2n} y(\sigma)\, e^{-c_2^{-1}(t-\sigma)}\, d\sigma \\ &\leqslant e^{-c_2^{-1}\cdot 2n}\int_{t-2(n+1)}^{t-2n} y(\sigma)\, d\sigma\leqslant Q_2 e^{-2nc_2^{-1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}I&\leqslant Q_2(1+e^{-2c_2^{-1}}+e^{-4c_2^{-1}}+\cdots+e^{-2[t/2]\,c_2^{-1}})\\ &<Q_2\frac{1}{1-e^{-2c_2^{-1}}}=Q_2\frac{e^{2c_2^{-1}}}{e^{2c_2^{-1}}-1}.\end{aligned}$$

Now apply Eq. (3.5); the conclusion of this lemma follows immediately. ∎

The following theorem states some sufficient conditions for the boundedness of the solutions of Eq. (1.5).

THEOREM 3.1. *In Eq.* (1.5), *assume that*

(i) $x(0)\, e^{-\int_0^t c^{-1}(s+1)\,ds}+Q_2 e^{2c_2^{-1}}/(e^{2c_2^{-1}}-1)\leqslant P(t)\, Q_1/Q_2$, *and*

(ii) $0\leqslant y(t)\leqslant Q_1$, *for* $-1\leqslant t\leqslant 0$, *and* $x(0)>0$.

Then

$$0<x(t)<x(0)\, e^{-\int_0^t c^{-1}(s+1)\,ds}+Q_2 e^{2c_2^{-1}}/(e^{2c_2^{-1}}-1)\qquad \textit{for}\quad t\geqslant 0.$$

Proof. By Lemma 2.1, $x(t)>0$ for all $t\geqslant 0$. Suppose that the conclusion of the theorem were false. Let $t^*>0$ be the least positive number such that

$$x(t^*)=x(0)\,e^{-\int_0^{t^*} c^{-1}(s+1)\,ds}+Q_2 e^{2c_2^{-1}}/(e^{2c_2^{-1}}-1), \tag{3.6}$$

and such that

$$0<x(t)<x(0)\,e^{-\int_0^t c^{-1}(s+1)\,ds}+Q_2 e^{2c_2^{-1}}/(e^{2c_2^{-1}}-1) \qquad \text{for} \quad 0\leqslant t<t^*.$$

Assumption (i) yields

$$0<x(t)\,r(t)\,c(t)\leqslant Q_1/Q_2 \qquad \text{for} \quad t\in[0,t^*].$$

Combining this with assumption (ii), and then applying Lemma 3.1 yields

$$0\leqslant y(t)\leqslant Q_1$$

and

$$\int_{\min\{0,t-2\}}^{t} y(s)\,ds\leqslant Q_2 \qquad \text{for} \quad t\in[0,t^*].$$

From Lemma 3.2, it is easy to see that

$$x(t)<x(0)\,e^{-\int_0^t c^{-1}(s+1)\,ds}+Q_2 e^{2c_2^{-1}}/(e^{2c_2^{-1}}-1) \qquad \text{for} \quad t\in[0,t^*].$$

This inequality obviously contradicts Eq. (3.6), which proves the theorem. ∎

This theorem indicates that as long as the two assumptions (i) and (ii) are met, the solution of Eq. (1.5) is bounded above by $x(0)+Q_2 e^{2c_2^{-1}}/(e^{2c_2^{-1}}-1)$ and, eventually, the solution is bounded above by $Q_2 e^{2c_2^{-1}}/(e^{2c_2^{-1}}-1)$.

4. Results in the Autonomous Case

When all the coefficient functions are constants, the following result is the counterpart of Theorem 3.1.

THEOREM 4.1. *Consider the equation*

$$\dot{x}(t)=rx(t)(a-x(t-1)-c\dot{x}(t-1)), \tag{4.1}$$

where r, a, and c are real constants and where $r>0$, $c>0$. In Eq. (4.1) *assume that*

$$\text{(i)}\quad x(0)\,e^{-t/c}+\frac{1}{c}\left(a+\frac{1}{rc}\right)e^{2/c}/(e^{2/c}-1)\leqslant\frac{1}{rc},$$

and that

$$\text{(ii)}\quad 0\leqslant\dot{x}(t)+c^{-1}x(t)\leqslant\frac{1}{c}\left(a+\frac{1}{rc}\right)\qquad \textit{for}\quad -1\leqslant t\leqslant 0 \textit{ and } x(0)>0.$$

Then,

$$0<x(t)<x(0)\,e^{-t/c}+\frac{1}{c}\left(a+\frac{1}{rc}\right)e^{2/c}/(e^{2/c}-1)\qquad \textit{for}\quad t\geqslant 0.$$

Proof. In this situation, $c(t)=c_1=c_2$, $p(t)=p=r^{-1}c^{-1}$, $Q(t)=Q_1=Q_2=(1/c)(a+1/rc)$ and $y(t)=\dot{x}(t)+c^{-1}x(t)$. The conclusion follows from Theorem 3.1. ∎

COROLLARY 4.1. *In Eq.* (4.1), *if*

(i) $x(0)>0$, $x(0)+\frac{1}{2}e^{1/c}(a+1/rc)<1/rc$, *and*

(ii) $0\leqslant\dot{x}(t)+c^{-1}x(t)\leqslant(1/c)(a+1/rc)$, *for* $-1\leqslant t\leqslant 0$,

then

$$0<x(t)<x(0)+\frac{1}{2}\,e^{1/c}\left(a+\frac{1}{rc}\right)\qquad \textit{for}\quad t\geqslant 0.$$

Proof. Since $e^{1/c}-e^{-1/c}\leqslant 2/c$, we have

$$e^{2/c}/(e^{2/c}-1)=e^{1/c}/(e^{1/c}-e^{-1/c})\leqslant\frac{c}{2}\,e^{1/c};$$

therefore

$$x(0)\,e^{-t/c}+\frac{1}{c}\left(a+\frac{1}{rc}\right)e^{2/c}/(e^{2/c}-1)\leqslant x(0)+\frac{1}{2}\,e^{1/c}\left(a+\frac{1}{rc}\right).$$

Hence conditions (i) and (ii) in Theorem 4.1 are met, which implies

$$\begin{aligned}0<x(t)&<x(0)\,e^{-t/c}+\frac{1}{c}\left(a+\frac{1}{rc}\right)e^{2/c}/(e^{2/c}-1)\\&<x(0)+\frac{1}{2}\,e^{1/c}\left(a+\frac{1}{rc}\right),\end{aligned}$$

for $t\geqslant 0$. ∎

EXAMPLE 4.1. In Eq. (4.1), let $r = 0.05$, $c = 4$, $a = 1$, $x(0) \leqslant 1$, and $x(t) = x(0)$, for $-1 \leqslant t \leqslant 0$. Then $x(0) + \frac{1}{2}e^{1/c}(a + 1/rc) \leqslant 4.9 < 1/rc = 5$, and

$$\dot{x}(t) + c^{-1}x(t) = \frac{1}{4}x(0) < \frac{1}{4} < \frac{1}{c}\left(a + \frac{1}{rc}\right) = \frac{3}{2} \qquad \text{for} \quad -1 \leqslant t \leqslant 0.$$

Therefore, by Corollary 4.1, $0 < x(t) < 4.9$, for all $t \geqslant 0$.

If $a < 0$ in Eq. (4.1), then the corresponding boundedness result is very simple, as the following result shows.

THEOREM 4.2. *If $a < 0$, $x(0) \leqslant -a$, $a + 1/rc > 0$, and $0 \leqslant \dot{x}(t) + c^{-1}x(t) \leqslant (1/c)(a + 1/rc)$, then*

$$0 < x(t) < \frac{1}{rc} \qquad \textit{for all} \quad t \geqslant 0.$$

Proof. By Lemma 2.1, $x(t) > 0$ for all $t \geqslant 0$. Suppose that the conclusion was false. Let $t^* > 0$ be the least positive number such that $x(t^*) = (rc)^{-1}$, and $0 < x(t) \leqslant (rc)^{-1}$ for $t \in [0, t^*]$. By Lemma 3.1, $0 \leqslant y(t) \leqslant (1/c)(a + 1/rc)$. Equation (3.5) implies that

$$\begin{aligned} x(t) &= x(0)\, e^{-(1/c)t} + e^{-(1/c)t} \int_0^t y(s)\, e^{(1/c)s}\, ds \\ &\leqslant x(0) + \frac{1}{c}\left(a + \frac{1}{rc}\right) e^{-(1/c)t} \int_0^t e^{(1/c)s}\, ds \\ &= x(0) + \left(a + \frac{1}{rc}\right)(1 - e^{-(1/c)t}) \\ &< x(0) + a + \tfrac{1}{rc} \leqslant \frac{1}{rc} \qquad \text{for} \quad t \in [0, t^*]. \end{aligned}$$

This obviously contradicts the assumption that $x(t^*) = (rc)^{-1}$, which proves the theorem. ∎

In [3], the following asymptotic result is obtained.

THEOREM 4.3 (H. I. Freedman and Y. Kuang [3]). *In Eq. (4.1), assume that r, a, c are positive constants. If $rac > 1$, then the steady-state solution $x(t) = a$ is unstable. If $rac < 1$, and*

$$(1 - r^2a^2c^2)^{1/2} \operatorname{arccot}(-rac(1 - r^2a^2c^2)^{-1/2}) > ra, \tag{4.2}$$

then $x(t) = a$ is asymptotically stable, while if the inequality (4.2) is reversed, then $x(t) = a$ is unstable.

It is interesting to compare this asymptotic conclusion in Theorem 4.3 with the results in this section. Consider the case $a = 1$, $rc = \frac{1}{5}$. Theorem 4.3 indicates that if r is very small, while c is very big, then (4.2) holds and hence the steady-state $x(t) = a$ is asymptotically stable. It turns out that the same assumption also yields the boundedness of its solution according to Theorem 4.1 or Corollary 4.1. Indeed, $r = 0.05$, $c = 4$, $x(0) \leqslant 1$, and $x(t) = x(0)$ for $-1 \leqslant t \leqslant 0$ together guarantee that $0 < x(t) < 4.9$, for all $t \geqslant 0$. Similarly for the case $r = 0.01$, $c = 20$, $x(0) \leqslant 1$, and $x(t) = x(0)$ for $-1 \leqslant t \leqslant 0$. In fact, if rc is fixed, then the bigger the constant c, the smaller the upper bound for $x(t)$ for $t \geqslant 0$. In other words, the results in this section agree with Theorem 4.3, which is obtained from [3].

5. Open Questions

This paper is devoted to the discussion of the boundedness of solutions of equation (1.5). Obviously, these results leave room for further improvement. There are several important problems related to Eq. (1.5) and the autonomous equation (4.1) remaining to be investigated. For example:

(1) What can be said about the existence of periodic solutions in Eq. (1.5) when the coefficient functions $r(t)$, $a(t)$ and $c(t)$ are periodic?

(2) For Eq. (4.1), what are the conditions for the global stability of the positive steady-state solution $x(t) = a$, in case of $a > 0$? A modified version of the method introduced in [7] may serve this purpose.

(3) The existence of Hopf bifurcation for Eq. (4.1) should be investigated. The integral manifold technique developed by J. K. Hale in [8, 9] may contribute to the solution of this interesting problem.

(4) What can be said if the delay in Eqs. (1.5) and (4.1) is distributed rather than a single discrete one?

The results stated in this paper can serve as steppingstones for future investigations of neutral delay interacting population models, such as predator–prey or competition systems.

References

1. K. L. Cooke and P. van den Driessche, On zeroes of some transcendental equations, *Funkcial. Ekvac.* **29** (1986), 77–90.
2. K. L. Cooke and Z. Grossman, Discrete delay, distributed delay and stability switches, *J. Math. Anal. Appl.* **86** (1982), 592–627.
3. H. I. Freedman and Y. Kuang, Stability switches in linear scalar neutral delay equations, *Funkcial. Ekvac.* **34** (1991), to appear.

4. K. Gopalsamy, On the global attractivity in a generalised delay logistic differential equation, *Math. Proc. Cambridge Philos. Soc.* **100** (1986), 183–192.
5. K. Gopalsamy, M. R. S. Kulenovic, and G. Ladas, Time lags in a "food-limited" population model, *Appl. Anal.* **31** (1988), 225–237.
6. K. Gopalsamy and B. G. Zhang, On a neutral delay-logistic equation, *Dynamics Stability Systems* **2** (1988), 183–195.
7. J. R. Haddock and J. Terjéki, Liapunov–Razumikhin functions and an invariance principle for functional differential equations, *J. Differential Equations* **48** (1983), 95–122.
8. J. K. Hale, Behavior near constant solutions of functional differential equations, *J. Differential Equations* **15** (1974), 278–294.
9. J. K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, Berlin, 1977.
10. J. K. Hale, "Asymptotic Behavior of Dissipative Systems," Mathematics Surveys and Monographs, American Mathematical Society, Providence, RI, 1988.
11. E. C. Pielou, "Mathematical Ecology," Wiley Interscience, New York, 1977.

Printed in Belgium